Journal of Statistical Physics, Vol. 77, Nos. 3/4, 1994

Topological and Fractal Properties of Turbulent Passive Scalar Fluctuations at Small Scales

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Received March 25, 1994

Corrections of Batchelor's spectral law "-1" of passive scalar fluctuations are obtained by taking into account the topological instabilities of small-scale vortex sheets: "$-4/3$" for supercritical and "$-5/4$" for subcritical regimes. The corresponding fractal dimensions of the scalar interface are $D_\sigma = 8/3$ for supercritical and $D_\sigma = 11/4$ for subcritical regimes. Good agreement with experimental data is established.

KEY WORDS: Helicity; vortex sheets; spectra; topological instabilities.

The problem of scaling laws of passive scalar fluctuations at the scales of the order of the Kolmogorov scale η has been related to the condition $\nu \gg \chi$, where ν is the molecular viscosity, and χ is the coefficient of molecular diffusion of a passive scalar. In this case, the field of velocity changes linearly with the coordinates, while the field of the passive scalar Θ has rather strong turbulent fluctuations. For this situation Batchelor obtained[1] the following scaling law for the spectral density E_Θ of passive scalar fluctuations:

$$E_\Theta(k) \propto \langle N \rangle \, \tau_\eta k^{-1} \tag{1}$$

where $\langle N \rangle = \chi \langle (\nabla \Theta)^2 \rangle$, $\tau_\eta = (\nu / \langle \varepsilon \rangle)^{1/2}$ (see also ref. 2).

In recent considerations[3-5] the fractal dimension of the scalar interface $D_\sigma = 3$ also has been obtained for this case. However, experimental observations described in these papers give another value of $D_\sigma \simeq 2.7$ for $r < \eta$. Figure 1 (taken from ref. 5) shows a plot of the logarithm of the num-

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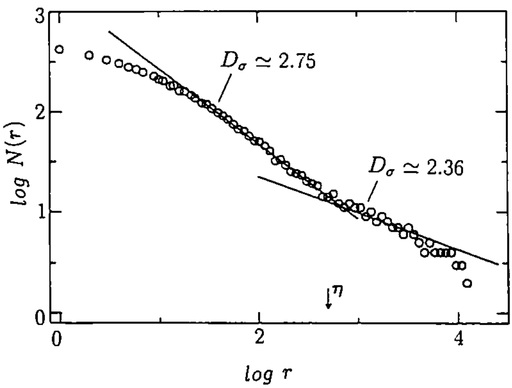

Fig. 1. A log–log plot of the number $N(r)$ of line elements of size r containing the interface versus the box size r in the viscous-convective interval of scales (the plane wake of a circular cylinder[3]).

ber of "boxes" containing the intersection points from one-dimensional cuts through the interface as a function of the box size. The data are from the wake experiment. The fractal dimension $D_\sigma \simeq 2.36$ is a characteristic value for the Kolmogorov inertial interval $r > \eta$,[5] and the value $D_\sigma \simeq 2.75$ is observed at $r < \eta$ (the viscous-convective interval[2]).

The authors of refs. 3–5 relate the difference between the observed values of D_σ (for $r < \eta$) and Batchelor's value $D_\sigma = 3$ to finite-Schmidt-number correction.

Another explanation of the observations can be obtained by taking account of local anisotropic singularities of the field of vorticity and helicity effects.[6–10]

Indeed, the simultaneous action of the linear velocity field and the viscous forces lead to the appearance of concentrated vortex sheets or lines at scales of the order of η.[2,6,7] The vortex sheets should be unstable in three-dimensional space and generally these instabilities have topological nature. The scaling laws pertaining to the secondary regimes (helical traveling waves) are governed by the topological parameter $\langle |dh/dt| \rangle$ [$h = (\mathbf{u} \operatorname{curl} \mathbf{u})$ is the helicity] in the case of *supercritical* instability, because of the spontaneous input of helicity into these helical waves[9,10] (for two-dimensional motion, $h = 0$). In the case of *subcritical* instabilities, the helical traveling waves are unstable and the secondary stable regime appears as the result of the modulation of these waves (in the self-focusing case, there appear helical solitons). Thus at the subcritical instabilities stable secondary regimes are governed by the modulation parameter: the rate of helicity variation $\langle |d^2h/dt^2| \rangle$ (cf. ref. 9). At the scales of the order of η, these parameters cannot be governing for the field of velocity because of the strong (direct) influence of the molecular viscosity, while for the field of the passive scalar they can.[8] In this case, we obtain the scaling spectral law

$$E_\Theta(k) \propto \langle N \rangle \langle |dh/dt| \rangle^{-1/3} k^{-4/3} \tag{2}$$

for supercritical regimes, and

$$E_\Theta(k) \propto \langle N \rangle \langle |d^2h/dt^2| \rangle^{-1/4} k^{-5/4} \tag{3}$$

for subcritical ones.

Figure 2 (adapted from ref. 4) shows the frequency spectrum of concentration fluctuations observed in the wake. The solid straight line is drawn for comparison with (2) (the Taylor hypothesis used). The difference between power laws (1) and (2) [and (3)] is hardly discernible in this spectrum. This is the common problem of all spectral measurements at these scales.

Vassilicos[11] linked the exponent in the power spectral law $E_\Theta(k) \propto k^{-n}$ to the fractal dimension D_σ (see also ref. 12)

$$D_\sigma = 4 - n \tag{4}$$

One can obtain from (2) and (4) $D_\sigma = 8/3$ for the supercritical case and $D_\sigma = 11/4 = 2.75$ for the subcritical regime (3). Thus we infer from Fig. 1 that the subcritical regime was taking place in this realization.

It may be interesting to note that in the upper atmosphere, quasi-two-

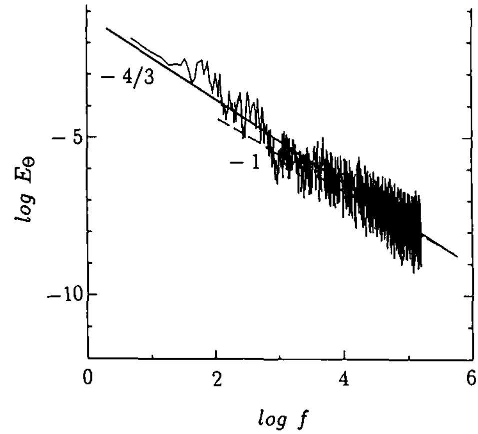

Fig. 2. The frequency spectrum of concentration fluctuations in the wake.[4]

dimensional turbulence forms at very large scales.[10] The reasons for quasi-two-dimensional geometry of motion in the atmosphere and in the viscous-convective interval of scales are very much different. However, the topological nature of the above scaling laws provides their universality. Figure 3 (adapted from ref. 13) shows horizontal spectra of atmospheric traces (ozone) measured in the stratosphere. The straight line is drawn for comparison with (2). Figure 4 (adapted from ref. 4) enables one to find the value of $D_p = D_\sigma - 1$ at analogous atmospheric conditions (cf. Figs. 1 and 2). The inversion in dispositions of the quasi-two-dimensional and the "Kolmogorov" scaling laws in Fig. 1 and 4 has quite clear reasons.

These instabilities are "large"-scale phenomena.[10,9] This means that they should work (if at all) in a large-scale part of the viscous-convective interval, while the Batchelor scaling (1) can be expected in a small-scale part of the interval (where the viscosity suppresses the instabilities). Since in this (small-scale) part of the interval the effect of molecular diffusion has already appeared, experimental observations of Batchelor's spectral *power* law (1) itself are very difficult.[2,15,16] It must, however, be remarked that multifractal measurements, unlike spectral ones, allow one to detach subregions with large values of a field.[9] Let us recall that the generalized dimension is $D_q \equiv 3$ for scaling (1), in accordance with refs. 3–5. The first multifractal measurements in the viscous-convective interval of scales have shown that this possibility is real.[5]

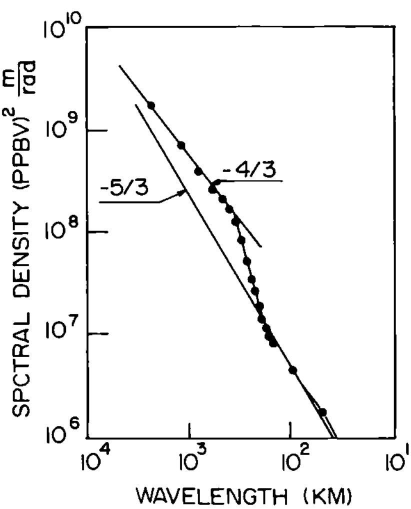

Fig. 3. Horizontal spectra of atmospheric traces (ozone). Adapted from ref. 13.

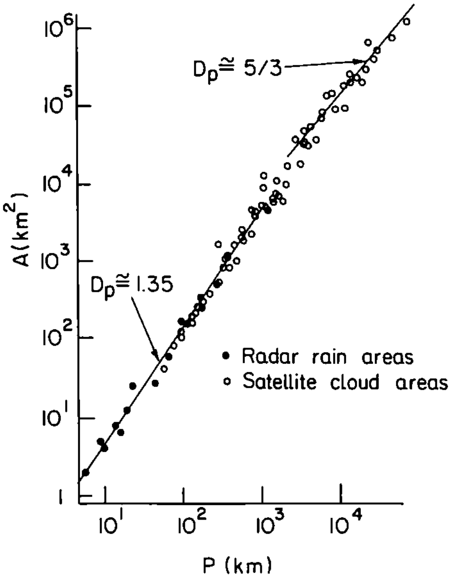

Fig. 4. Scaling relations of area A versus perimeter for clouds, allowing one to obtain the dimension $D_p = D_\sigma - 1$. Adapted from ref. 14.

ACKNOWLEDGMENTS

The author is grateful to L. Kadanoff, S. Lovejoy, C. Meneveau, H. K. Moffatt, K. R. Sreenivasan, A. Tsinober, N. Tsitverblit, J. C. Vassilicos, and a referee for information, comments, and encouragement at different stages of this work.

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Communicated by J. L. Lebowitz